M2: příklady sestavené na základě zkouškových písemek

Dána funkce $f(x,y)$ a
bod dotyku $P = [x_0, y_0, z_0 = f(x_0, y_0)]$, bod $A \in \mathcal{D}(f)$ a směr $\vec{s}$.

1. Popište a načrtněte v $\mathbb{E}_2$ množinu $\mathcal{M}$, kde je funkce diferencovatelná.
2. Určete parciální derivace $\frac{\partial f}{\partial x}$ a $\frac{\partial f}{\partial y}$.
3. Napište rovnici tečné roviny ke grafu funkce v bodě dotyku P.
   Zapište parametrické rovnice normály.
4. Vypočítejte derivaci funkce f v bodě A ve směru $\vec{s}$.

poznámka:
rovnice tečné roviny ke grafu funkce $z = f(x,y)$ v bodě dotyku P= $[x_0, y_0, z_0]$:
$\tau : z - z_0 = \frac{\partial f}{\partial x}(P)(x - x_0) + \frac{\partial f}{\partial y}(P)(y - y_0),\ z_0 = f(x_0, y_0)$,
derivace funkce f v bodě A ve směru $\vec{s}$:

$$\frac{\partial f}{\partial \vec{s}}(A) = \frac{\text{grad } f(A) \cdot \vec{s}}{\|\vec{s}\|}$$

1. $f(x,y) = \sqrt{x^2 - y^2}, \quad P = [5, 3, ?], \quad A = [5, 3], \quad \vec{s} = (1, 1)$

$$\mathcal{M} = \{[x, y] \in \mathbb{E}_2 : |x| > |y|\}$$

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 - y^2}}, \frac{\partial f}{\partial y} = -\frac{y}{\sqrt{x^2 - y^2}}$$

$$\tau : z - 4 = \frac{5}{4}(x - 5) - \frac{3}{4}(y - 3), \quad n : \begin{array}{l} x = 5 + \frac{5}{4}t \\ y = 3 - \frac{3}{4}t \\ z = 4 - t \end{array}, \ t \in \mathbb{R}$$

$$\frac{\partial f}{\partial \vec{s}}(A) = \frac{(\frac{5}{4}, -\frac{3}{4}) \cdot (1, 1)}{\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

2. $f(x,y) = \sqrt{5y - x^2}, \quad P = [3, 5, ?], A = [3, 5], \quad \vec{s}$ je směrem nejvyššího růstu funkce f v bodě A.

$$\mathcal{M} = \left\{[x, y] \in \mathbb{E}_2 : y > \frac{x^2}{5}\right\}$$

$$\frac{\partial f}{\partial x} = -\frac{x}{\sqrt{5y - x^2}}, \quad \frac{\partial f}{\partial y} = \frac{5}{2\sqrt{5y - x^2}}, \text{ grad } f(A) = \left(-\frac{3}{4}, \frac{5}{8}\right)$$

$$\tau : z - 4 = -\frac{3}{4}(x - 3) + \frac{5}{8}(y - 5), \quad n : \begin{array}{l} x = 3 - \frac{3}{4}t \\ y = 5 + \frac{5}{8}t \\ z = 4 - t \end{array}, \ t \in \mathbb{R}$$

$$\vec{s} = \text{grad } f(A) = \left(-\frac{3}{4}, \frac{5}{8}\right)$$

$$\frac{\partial f}{\partial \vec{s}}(A) = \frac{\sqrt{61}}{8}$$

3. $f(x,y)=\sqrt{x^2+y^2-9},\quad P=[4,3,?],\quad A=[4,3],\quad \overrightarrow{s}=(1,2)$

$$\mathcal{M}=\{[x,y]\in\mathbb{E}_2 : x^2+y^2>9\}$$

$$\frac{\partial f}{\partial x}=\frac{x}{\sqrt{x^2+y^2-9}},\ \frac{\partial f}{\partial y}=\frac{y}{\sqrt{x^2+y^2-9}},\ \text{grad } f(A)=\left(1,\frac{3}{4}\right)$$

$$\tau : z-4=(x-4)+\frac{3}{4}(y-3),\quad n:\ \begin{array}{l} x=4+t \\ y=3+\frac{3}{4}t \\ z=4-t \end{array},\ t\in\mathbb{R}$$

$$\frac{\partial f}{\partial \overrightarrow{s}}(A)=\frac{\sqrt{5}}{2}$$

4. $f(x,y)=\sqrt{x-y^2},\quad P=[5,-1,?],\quad A[5,-1,]\quad \overrightarrow{s}$ je směrem nejvyššího růstu funkce f v bodě A

$$\mathcal{M}=\{[x,y]\in\mathbb{E}_2 : x>y^2\}$$

$$\frac{\partial f}{\partial x}=\frac{1}{2\sqrt{x-y^2}},\ \frac{\partial f}{\partial y}=-\frac{y}{\sqrt{x-y^2}},\text{grad } f(A)=\left(\frac{1}{4},\frac{1}{2}\right)$$

$$\tau : z-2=\frac{1}{4}(x-5)+\frac{3}{4}(y+1),\quad n:\ \begin{array}{l} x=5+\frac{1}{4}t \\ y=-1+\frac{3}{4}t \\ z=2-t \end{array},\ t\in\mathbb{R}$$

$$\overrightarrow{s}=\text{grad } f(A),\ \frac{\partial f}{\partial \overrightarrow{s}}(A)=\frac{\sqrt{5}}{4}$$

5. $f(x,y)=\sqrt{9-x^2-y^2},\quad P=[-1,2,?],\quad A=[-1,2],\quad \overrightarrow{s}$ je směrem nejvyššího poklesu funkce f v bodě A

$$\mathcal{M}=\{[x,y]\in\mathbb{E}_2 : x^2+y^2<9\}$$

$$\frac{\partial f}{\partial x}=-\frac{x}{\sqrt{9-x^2-y^2}},\ \frac{\partial f}{\partial y}=-\frac{y}{\sqrt{9-x^2-y^2}},\ \text{grad } f(A)=\left(\frac{1}{2},-1\right)$$

$$\tau : z-2=\frac{1}{2}(x+1)-(y-2),\quad n:\ \begin{array}{l} x=-1+\frac{1}{2}t \\ y=2-t \\ z=2-t \end{array},\ t\in\mathbb{R}$$

$$\overrightarrow{s}=-\text{grad } f(A)=\left(-\frac{1}{2},1\right),\ \frac{\partial f}{\partial \overrightarrow{s}}(A)=-\frac{\sqrt{5}}{2}$$

6. $f(x,y)=\sqrt{y-x^2},\quad P=[-2,5,?],\quad A=[-2,5],\quad \overrightarrow{s}$ je směrem nejvyššího růstu funkce f v bodě A

$$\mathcal{M}=\{[x,y]\in\mathbb{E}_2 : y>x^2\}$$

$$\frac{\partial f}{\partial x}=-\frac{x}{\sqrt{y-x^2}},\ \frac{\partial f}{\partial y}=\frac{1}{2\sqrt{y-x^2}},\ \text{grad } f(A)=\left(2,\frac{1}{2}\right)$$

$$\tau : z-1=2(x+2)+\frac{1}{2}(y-5),\quad n:\ \begin{array}{l} x=-2+2t \\ y=5+\frac{1}{2}t \\ z=1-t \end{array},\ t\in\mathbb{R}$$

$$\overrightarrow{s}=\text{grad } f(A),\ \frac{\partial f}{\partial \overrightarrow{s}}(A)=\frac{\sqrt{17}}{2}$$

7. $f(x,y) = \ln(xy-4)$, $P = [-2,-4,?]$, $A = [-2,-4]$, $\overrightarrow{s}$ je směr, ve kterém je derivace funkce f v bodě A nulová. Určete $\overrightarrow{s}$.

$$\mathcal{M} = \{[x,y] \in \mathbb{E}_2 : xy > 4\}$$

$$\frac{\partial f}{\partial x} = \frac{y}{xy-4},\ \frac{\partial f}{\partial y} = \frac{x}{xy-4},\ \text{grad } f(A) = \left(-1, -\frac{1}{2}\right)$$

$$\tau : z - \ln 4 = -(x+2) - \frac{1}{2}(y+4), \quad n : \begin{array}{l} x = -2-t \\ y = -4 - \frac{1}{2}t \\ z = \ln 4 - t \end{array},\ t \in \mathbb{R}$$

$$\frac{\partial f}{\partial \vec{s}}(A) = 0 \Leftrightarrow (-1, -\frac{1}{2}) \cdot (s_x, s_y) = 0\ ;\ -s_x - \frac{1}{2}s_y = 0,\ \vec{s} = (t, -2t), t \in \mathbb{R},\ t \neq 0,$$

např. $\vec{s} = (1, -2)$

8. $f(x,y) = \ln(3 - x^2 - y^2)$, $P = [1,1,?]$, $A = [1,1]$, $\vec{s}$ je směr, ve kterém je derivace funkce f v bodě A nulová. Určete $\vec{s}$.

$$\mathcal{M} = \{[x,y] \in \mathbb{E}_2 : x^2 + y^2 < 3\}$$

$$\frac{\partial f}{\partial x} = \frac{-2x}{(3-x^2-y^2)}\ \frac{\partial f}{\partial y} = \frac{-2y}{(3-x^2-y^2)},\ \text{grad } f(A) = (-2,-2)$$

$$\tau : z = -2(x-1) - 2(y-1), \quad n : \begin{array}{l} x = 1-2t \\ y = 1-2t \\ z = -t \end{array},\ t \in \mathbb{R}$$

$$\frac{\partial f}{\partial \vec{s}}(A) = 0 \Leftrightarrow (-2,-2) \cdot (s_x, s_y) = 0\ ;\ -2s_x - 2s_y = 0,\ \vec{s} = (t, -t), t \in \mathbb{R},\ t \neq 0,$$

např. $\vec{s} = (1, -1)$

9. $f(x,y) = \ln(xy-2)$, $P = [-3,-1,?]$, $A[-3,-1]$, $\vec{s}$ je směr, ve kterém je derivace funkce f v bodě A nulová. Určete $\vec{s}$.

$$\mathcal{M} = \{[x,y] \in \mathbb{E}_2 : xy > 2\}$$

$$\frac{\partial f}{\partial x} = \frac{y}{xy-2},\ \frac{\partial f}{\partial y} = \frac{x}{xy-2},\ \text{grad } f(A) = (-1,-3)$$

$$\tau : z = -(x+3) - 3(y+1), \quad n : \begin{array}{l} x = -3-t \\ y = -1-3t \\ z = -t \end{array},\ t \in \mathbb{R}$$

$$\frac{\partial f}{\partial \vec{s}}(A) = 0 \Leftrightarrow (-1,-3) \cdot (s_x, s_y) = 0\ ;\ -s_x - 3s_y = 0,\ \vec{s} = (-3t, t), t \in \mathbb{R},\ t \neq 0,$$

např. $\vec{s} = (-3, 1)$

10. $f(x,y) = \ln(x^2+y^2-4)$, $P = [2,1,?]$, $A = [2,1]$, $\vec{s}$ je směrem nejvyššího poklesu funkce f v bodě A

$$\mathcal{M} = \{[x,y] \in \mathbb{E}_2 : x^2 + y^2 > 4\}$$

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2-4},\ \frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2-4},\ \text{grad } f(A) = (4,2)$$

$$\tau : z = 4(x-2) + 2(y-1), \quad n : \begin{array}{l} x = 2+4t \\ y = 1+2t \\ z = -t \end{array},\ t \in \mathbb{R}$$

$$\vec{s} = -\text{grad } f(A) = (-4,-2),\ \frac{\partial f}{\partial \vec{s}}(A) = -2\sqrt{5}$$

11. $f(x,y) = \ln(x-y^2)$, $P=[2,1,?]$, $A=[2,1]$, $\vec{s}$ je směrem nejvyššího růstu funkce f v bodě A

$$\mathcal{M} = \{[x,y] \in \mathbb{E}_2 : x > y^2\}$$

$$\frac{\partial f}{\partial x} = \frac{1}{(x-y^2)},\ \frac{\partial f}{\partial y} = -\frac{2y}{(x-y^2)},\ \text{grad } f(A) = (1,-2)$$

$$\tau : z = (x-2) - 2(y-1), \quad n: \begin{array}{l} x = 2+t \\ y = 1-2t \\ z = -t \end{array},\ t \in \mathbb{R}$$

$$\vec{s} = \text{grad } f(A),\ \frac{\partial f}{\partial \vec{s}}(A) = \sqrt{5}$$

12. $f(x,y) = \ln(y-3x)$, $P=[1,5,?]$, $A=[1,5]$, $\vec{s}=(1,1)$

$$\mathcal{M} = \{[x,y] \in \mathbb{E}_2 : y > 3x\}$$

$$\frac{\partial f}{\partial x} = -\frac{3}{y-3x},\ \frac{\partial f}{\partial y} = \frac{1}{y-3x},\ \text{grad } f(A) = \left(-\frac{3}{2}, \frac{1}{2}\right)$$

$$\tau : z - \ln 2 = -\frac{3}{2}(x-1) + \frac{1}{2}(y-5), \quad n: \begin{array}{l} x = 1-\frac{3}{2}t \\ y = 5+\frac{1}{2}t \\ z = \ln 2 - t \end{array},\ t \in \mathbb{R}$$

$$\frac{\partial f}{\partial \vec{s}}(A) = -\frac{\sqrt{2}}{2}$$